The Principle of Equivalence and Electro-magnetism.

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The problem of unification of electro-magnetism and gravitation in four dimensions; some new ideas involving the use of mixtures of commuting and anti-commuting co-ordinates. Maxwell' s equations are extracted in terms of curvature of the anti-commuting part of space-time. The profound difference in the coupling constants of the two forces is interpreted in terms of the degree of expansion of the two kinds of space-time with evolution of the universe. Uncertainty in the quantum realm is interpreted in terms of an unmeasurable component of anti-commuting space-time.

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I. INTRODUCTION

The world of experience suggests to us that space-time is a real continuum which may be represented in terms of systems of commuting co-ordinates composed of real numbers. It is thus natural that we should extrapolate this experience to the world of quantum objects and seek to describe them as embedded in a real commuting space-time. However, the quantum domain is distinctly different from our macroscopic experience so that one is led to question whether this extrapolation is wholly appropriate.

In seeking an alternative to standard commuting space-time one is led, by means of the principle of relativity, to consider whether there exists more general forms of co-ordinates for which the laws of nature remain valid. One favored means of further 'generalizing' space-time is to consider systems of commuting co-ordinates of dimension higher than four [11], [5]. However, another possibility which might be considered is that, in addition to systems of commuting co-ordinates, one may have systems of anti-commuting co-ordinates (or more generally arbitrary mixtures of commuting and anti-commuting co-ordinates). If anti-commuting co-ordinates exist in the quantum domain then one may interpret their apparent absence in the macroscopic domain of experience in much the same way as one considers the absence of macroscopic spinorial objects; over macroscopic distances the anti-commuting co-ordinates effectively 'cancel-out'.

We as macroscopic observers cannot conceive of a space-time of anti-commuting co-ordinates for this is outside our domain of experience. The world of macroscopic

objects is in a sense purely 'bosonic' for the rotation by 2π of any macroscopic object around any (single) arbitrary axis brings it into self-congruence. By contrast the wave function of a spinorial object changes sign with a rotation by 2π; an indication that, if an observer could become 'spinorial', the observers experience of space-time would be very different to that normally expected. In standard theory spinors such as electrons are represented by embedding them in a standard commuting Lörentz space-time. This mathematical representation works perfectly well; although such success should not be taken to indicate that such an embedding is necessarily both a complete and correct description of space-time. One hint that it may in fact be incomplete comes from the failure of attempts to combine General Relativity with quantum theory whereupon difficulties of an extreme sort are encountered such as the Coleman-Mandula (C.M.) theorem [1] which forbids the (non-trivial) union of compact and non-compact groups.

Is it possible that our extrapolation of commuting space-time to the quantum domain, in spite of its naturalness and success in the standard model, is incorrect or incomplete? Is it possible that our interpretation of the nature of spinorial matter is colored by our preference for commuting space-time for, at least at the intuitive level, one might guess that a spinorial object such as an electron will be 'bosonic' in anti-commuting space-time? Is it possible to avoid the C.M. theorem in mixed commuting and anti-commuting co-ordinates? (Supersymmetry, a symmetry connecting bosons and fermions, is a known way of avoiding the C.M. theorem [2]).

Putting aside for a moment the mathematical difficulties involved in mixing commuting and anti-commuting co-ordinates consider as an introduction to the

premise of this paper the following. Let us speculate that the universe began as a mixture of commuting and anti-commuting co-ordinate space time in a roughly one-to-one mixture. We might then further speculate that, in keeping with the big-bang model, the commuting part of space-time then expands over time but the anti-commuting part of space-time is far less prone to expansion because of a tendency of the anti-commuting space-time to 'cancel-out' over macroscopic distances with inflation with the effective fixed 'range' of the anti-commuting co-ordinate space-time being set by virtue of the quantisation of charge.

If the curvature of the commuting part of space-time then becomes the manifestation of gravitation does there exist an analogue of curvature of the anti-commuting part of space-time and what does it represent? It is the purpose of this paper to explore these and related questions. Several things immediately emerge to give us an indication of the possible answers.

The first is that, clearly, the curvature of the anti-commuting space-time must relate to a massless gauge field in much the same way as gravitation will relate to a massless gauge field. The reason is that the anti-commuting space-time is not confined as such; its macroscopic effects are attenuated by cancelations rather than an abrupt finite distance limit and thus its associated boson must be massless for potential infinite range. There is only one known (non-gravitation) candidate field quanta for this and it is the photon.

The second is that we may expect the coupling strength ratio of the two massless gauge fields (one for the commuting co-ordinates and one for the anti-commuting co-ordinates) to evolve as a function of effective relative expansion; beginning at ap-

proximately the time of the big-bang with a ratio of order unity. This suggests that the coupling constant of gravity is a dynamical quantity related to the size and/or age of the universe whereas the coupling constant of electro-magnetism evolves as a function of the *effective* range of the anti-commuting co-ordinates; the anti-commuting space-time must have stretched only minimally if at all since the big-bang giving us an explanation for the vast difference in the relative strength of the couplings of gravitation and electromagnetism in the current epoch.

To form the appropriate non-inertial frame to model the electro-magnetic field for the electron we must have co-ordinates capable of describing the space it is embedded in; since an electron is a spinor the possibility that a non-inertial anti-commuting frame may be appropriate for this task warrants consideration. In this we can see an analogue with general relativity. Inertial co-ordinates are not adequate for a description of the laws of nature for, from all possible co-ordinate systems, they single out one preferred frame; the laws of nature should be valid in all frames. Likewise, the assumption of purely commuting co-ordinates is unacceptable since the laws of nature should be valid under more general co-ordinates which may include some component of anti-commuting co-ordinates or in general an arbitrary mixture of commuting and anti-commuting co-ordinates.

In this way we hope to see electro-magnetism and gravitation as two sides of one space-time. At a fundamental level space-time will then be constructed of a continuum which has both commuting and anti-commuting properties. Curvature of the commuting aspect of space-time is purely additive an leads to macroscopic fields associated with macroscopic objects such as stars. Curvature of the anti-commuting

aspect of space-time then must lead to electric fields of particles and their spin-induced magnetic moments. The inherently dual nature of anti-commuting space time leads to dual types of fields; fields of attractive and repulsive force which, over macroscopic distances involving macroscopic matter, have a tendency to cancel-out. The free electro-magnetic field will then be interpreted as a propagating disturbance in the space-time vacuum due to temporary fluctuation away from a net 'zero' anti-commuting co-ordinates in regions distant from charged sources.

To represent electro-magnetism as curvature we need to extend the principle of equivalence. If we wish to describe an object accelerating in space-time we may choose to do so with respect to an inertial frame of reference. The object in question will then not be traveling with constant velocity with respect to such a frame. However, we may alternatively choose a non-inertial frame of reference - curved co-ordinates - with respect to which, provided the form of the non-inertial co-ordinates is appropriately chosen, our accelerating object describes a path of constant velocity. This is just the statement that a non-inertial frame is non-inertial i.e. accelerating. It was the recognition of this simple fact that led Einstein to G.R. for he realized that, considering an object 'at rest' on the surface of the Earth as an accelerating object but one moving with constant velocity, one is immediately let to deduce that the co-ordinates of the space are consequently not inertial. The specific form of these non-inertial co-ordinates then must describe the gravitational field causing the 'acceleration'. This is the principle of equivalence; most succinctly put in terms of the equivalence of inertial and gravitational mass [6], [21].

How might this principle be extended to include electro-magnetism; a force which

bears much similarity to gravitation as both are gauge fields and both are mediated by massless bosons (if one accepts the reality of gravitons)? Let us consider an electron's charge as the analogue of the mass which is the basis of the principle of equivalence in gravitation theory. One of the problems posed at the beginning of this century by the atomic model was 'why doesn't the electron, since it is accelerated whilst it orbits the nucleus, radiate energy and spiral into the positively charged nucleus?' The answer given was that the energy of the electron, and any energy it radiated, was quantized and could only come in quantum 'jumps' not a continuum.

However there is another way of looking at this problem in terms of the above postulate. If we consider a charged object such as an electron moving with constant velocity in the vacuum we may think of the surrounding electric field of the electron as approximately a symmetric (in terms of the direction of motion) finite radius of anti-commuting space-time (by 'finite' it is implied that the field is ≈ 0 at macroscopic distances). The effective radius of the field, which we may define at some arbitrary strength cut-off, may be taken as a constant for an inertial observer independent of where the cut-off is defined. However if the charge is *accelerating* with respect to an inertial observer an asymmetry will develop in the field between the forward direction (of motion) and backward direction anti-commuting space-time and this may be interpreted in terms of generated waves in the anti-commuting space-time; photons in this schema. That an electron in a hydrogen atom does not radiate energy then implies that the electron is traveling only in a straight line with constant velocity in terms of anti-commuting space-time and so does not radiate energy; the anti-commuting space must then be curved. The quantized nature of

the possible paths for the electron then arises because the analogue of mass - in this case the charge of the particle - comes in quantum units not in a continuum (the possible masses of macroscopic objects may be regarded as a continuum).

Thus we shall seek to recast general relativity in these more generalized co-ordinates. In so doing we shall seek to extend the principle of equivalence to encompass the concept of curvature in anti-commuting space-time as a representation of the electro-magnetic field. This principle will be expressed as the vanishing of the covariant derivative of the metric; an expression that a frame exists *locally* in anti-commuting co-ordinates in which the field can be 'transformed away'; the analogue of a free-fall frame.

As an introduction consider the (Lörentz covariant) Gupta-Bleuler formalism for photon quantization in which we have the following commutator;

$$[a^{\alpha}(k), a^{\dagger\beta}(k')]_{-} = -g_s^{\alpha\beta}\delta^3(k-k')$$

where $g_s^{\alpha\beta}$ is the standard symmetric metric diag.$(+,-,-,-)$ with the suffix 's' indicating symmetric. Using $g^s_{\alpha\beta}g_s^{\beta\alpha} = +4$ this is rewritten;

$$[a^{\alpha}(k), a^{\dagger}{}_{\alpha}(k')]_{-} = -4\delta^3(k-k') \tag{1}$$

However if $g^{\alpha\beta} \equiv g_a^{\alpha\beta}$ is purely *antisymmetric* and if it also has the appropriate mathematical properties to raise and lower the indices of the creation and annihilation operators $a^{\dagger}$ and a then its substitution into the commutator produces;

$$\begin{aligned} -\bar{g}^a_{\alpha\beta}[a^{\alpha}(k), a^{\dagger\beta}(k')]_{-} &= a_{\beta}(k).a^{\dagger\beta}(k') + a^{\dagger}_{\alpha}(k').a^{\alpha}(k) \qquad (2) \\ &= \{a^{\alpha}(k), a^{\dagger}_{\alpha}(k')\}_{+} \\ &= c.\delta^3(k-k') \end{aligned}$$

where $\bar{g}^a_{\alpha\beta}$ is the complex-conjugate transpose or 'dual' metric and for some constant $c = \bar{g}^a_{\alpha\beta} g^{\beta\alpha}_a$ (provided $c \neq 0$) a commutator has been converted to an anti-commutator; i.e. the interchange of a symmetric and anti-symmetric metric has implied an interchange of particle description from spin 1 to $\frac{1}{2}$; this implies that the particle statistics have changed! This is of course inconsistent with observation. However, if we add, in addition to the substitution of an anti-symmetric metric anti-commuting co-ordinates then we may suspect that the proper statistics will be restored. Thus we may suspect that general admixtures of symmetric and anti-symmetric metric, if appropriately combined with mixtures of commuting and anti-commuting co-ordinates, is covertly 'supersymmetric' if the laws of nature are invariant with respect to whatever arbitrary admixture is chosen. This approach however differs fundamentally from conventional supersymmetry in that no super-partner particles ever arise.

This simple exercise gives some hope that by modifying the structure of space-time a property somewhat analogous to supersymmetry, with the concomitant hope of avoidance of the C.M. theorem, might be embedded into the structure of space-time itself. In the next sections mathematical machinery is developed to give effect to such a structure.

II. METRIC AND CO-ORDINATES.

Our first task it find the form of the metric and the co-ordinates appropriate for an 'inertial frame' in anti-commuting space-time. We must then seek to extend the principle of equivalence to non-inertial anti-commuting co-ordinate systems. The

introduction of anti-commuting co-ordinates means that the metric will also have to be modified to include an anti-symmetric component; this is so that the invariant length

$$ds^2 = \delta x_a^\alpha g^s_{\alpha\beta} \delta x_a^\beta + \delta x_a^\alpha g^a_{\alpha\beta} \delta x_a^\beta$$

is well defined in anti-commuting co-ordinates. Here s = symmetric and a = anti-symmetric (so that $x_a^\alpha x_a^\beta = -x_a^\beta x_a^\alpha \ ; \ \forall \alpha \neq \beta$); the co-ordinates only commute if they have the same index (this is required so that $x_a^\alpha x_\alpha^a \neq 0$). If the combined metric $g_{\alpha\beta} = g^s_{\alpha\beta} + g^a_{\alpha\beta}$ is an invariant tensor in the theory then ds^2 is also an invariant.

Consider the following metric which is a 16x16 matrix but has only four space-time indices;

$$\frac{1}{2}\begin{pmatrix} -I_4 & i\sigma_{01} & i\sigma_{02} & i\sigma_{03} \\ i\sigma_{10} & +I_4 & i\sigma_{12} & i\sigma_{13} \\ i\sigma_{20} & i\sigma_{21} & +I_4 & i\sigma_{23} \\ i\sigma_{30} & i\sigma_{31} & i\sigma_{32} & +I_4 \end{pmatrix} \equiv -\frac{1}{2} I_4.\eta_{\alpha\beta} + \frac{i}{2}\sigma_{\alpha\beta} \tag{3}$$

where $\eta_{\alpha\beta}$ =diag.$(+,-,-,-)_{\alpha\beta}$ and $\sigma^{\alpha\beta} = \frac{i}{2}\left[\gamma^\alpha, \gamma^\beta\right]$. As a consequence of the choice of $\eta_{\alpha\beta}$ the gamma matrices follow the form $\gamma^{\dagger 0} = \gamma_0 = \gamma^0$ and $\gamma^{\dagger i} = \gamma_i = -\gamma^i$. This metric has the following closure property;

$$\begin{aligned} g_{\alpha\phi}\bar{g}^{\phi}{}_{\beta} &= \frac{1}{4}\left(-I_4.\eta_{\alpha\phi} + i\sigma_{\alpha\phi}\right)\left(-I_4.\eta^{\phi}{}_{\beta} - i\sigma^{\phi}{}_{\beta}\right) \\ &= \frac{1}{2}\left(-I_4.\eta_{\alpha\beta} + i\sigma_{\alpha\beta}\right) = g_{\alpha\beta} \end{aligned} \tag{4}$$

where I have used $\gamma^0\gamma_0 = \gamma^1\gamma_1 = \gamma^2\gamma_2 = \gamma^3\gamma_3 = I_4$ [1] and $\bar{g}$ is the ‘dual’ metric viz;

[1]most Q.F.T. textbooks contain the relation $\gamma^\alpha\gamma_\alpha = 4$ with a summation convention on

$$(\sigma_{\alpha\beta})^{\dagger} = \frac{-i}{2}\left[\gamma_{\alpha}, \gamma_{\beta}\right]^{\dagger} = \sigma^{\alpha\beta} \tag{5}$$

viz $\gamma^{\dagger 0} = \gamma_0 = \gamma^0$ and $\gamma^{\dagger i} = \gamma_i = -\gamma^i$ with $\eta_{\alpha\beta} = \eta^{\alpha\beta}$ a purely real diagonal matrix so that;

$$(g_{\alpha\beta})^{\dagger} = \bar{g}^{\alpha\beta} = \frac{1}{2}\left(-I_4.\eta^{\alpha\beta} - i\sigma^{\alpha\beta}\right) = g^{\beta\alpha} \tag{6}$$

The 'bar' operation is represented by taking the complex-conjugate transpose and raising (lowering) all indices. Using a complex metric such as metric.(3) requires generalization of the Lörentz transformation [2]. Under the modified Lörentz transformation the metric transforms as an invariant tensor as is required for a metric;

$$g_{\mu'\nu'} = \bar{\Lambda}_{\mu'}^{\ \mu} g_{\mu\nu} \Lambda_{\nu'}^{\ \nu} \tag{7}$$

There is a further complication concerning the use of metric.(3) as a space-time metric. Consider the following transformation of the co-ordinates under the modified Lörentz transformation given in the appendix;

$$\left(x^{\alpha'}\right)^{\dagger} = \left(\bar{\Lambda}^{\alpha'}{}_{\alpha} x^{\alpha}\right)^{\dagger} = x^{\dagger\alpha} \bar{\Lambda}^{\dagger\alpha'}{}_{\alpha} = x^{\dagger\alpha} \bar{\Lambda}^{\alpha}{}_{\alpha'} \tag{8}$$

where the last step follows from the property of the metric under complex-conjugate transpose eq.(6). Thus we require that the co-ordinates have the property $x^{\dagger\alpha} = x_{\alpha}$ i.e. the co-ordinates must act under complex-conjugation like the gamma matrices.

Now this can be achieved because of the following property of metric (3);

α but strictly speaking the gamma matrices square up to a 4x4 identity matrix not a scalar in which case $\gamma^{\alpha}\gamma_{\alpha} = 4I_4$.

[2]see appendix A.

$$\bar{g}_{\alpha\beta}\frac{\gamma^{\beta}}{2} = \frac{\gamma_{\alpha}}{2} \quad \text{and} \quad g_{\alpha\beta}\frac{\gamma^{\dagger\beta}}{2} = \frac{\gamma_{\alpha}}{2} \tag{9}$$

which, for the chosen representation $\gamma^{\dagger 0} = \gamma_0, \;\; \gamma^{\dagger i} = \gamma_i$, is a property one normally only expects to find in the purely symmetric diagonal metric $\eta = (+, -, -, -)$. In order that the constraint $x^{\dagger\alpha} = x_\alpha$ is satisfied it is necessary to couple each space-time co-ordinate to a gamma matrix so that, for example, the scalar x^0 gets multiplied into each entry of a γ^0 matrix and becomes $\frac{1}{2}x^0.\gamma^0 \equiv x^{\not 0}$ and $\frac{1}{2}x^1.\gamma^1 \equiv x^{\not 1}$ etc. From now on slashed $x^{\not\alpha}$ means space-time co-ordinates coupled to the corresponding gamma matrix; i.e. each co-ordinate is now represented by a 4x4 matrix. To indicate that the corresponding 16x16 metric is intended to contract against these modified co-ordinates the metric indices are also labeled with a slash i.e.

$$g_{\not\alpha\not\beta} \equiv \frac{-I_4}{2}\eta_{\alpha\beta} + \frac{i}{2}\sigma_{\alpha\beta}.$$

It is important to realize that the information as to whether a co-ordinate index is 'upstairs' or 'downstairs' is now carried on the gamma matrix index *not* on the co-ordinate scalar so for the scalars $x^0 = x_0, x^1 = x_1$ etc. whilst for the matrices $x^{\not 0} = x_{\not 0}, x^{\not 1} = -x_{\not 1}$ etc.

Note that the invariant length ds^2 is now positive definite (but only if translated into terms of commuting co-ordinates); i.e. related to a compact group;

$$\begin{aligned} ds^2 &= dx^{\not\alpha}\bar{g}_{\not\alpha\not\beta}dx^{\not\beta} \\ &= \frac{1}{4}(dx^0\gamma^0\gamma_0 dx_0 + dx^1\gamma^1\gamma_1 dx_1 + dx^2\gamma^2\gamma_2 dx_2 + dx^3\gamma^3\gamma_3 dx_3) \\ &= dx^0 dx_0 + dx^i dx_i \end{aligned} \tag{10}$$

where the last step follows because ds is strictly a scalar; that is, the trace must be

taken over the 4x4 identity matrices which result from the product of each pair of gamma matrices i.e. $Tr\gamma^0\gamma_0 = TrI_4 = 4$ etc. But note that the Lörentz structure is still manifest if we keep the anti-commuting structure;

$$ds^2 = dx^{\not\alpha} dx_{\not\alpha} = (dx^{\not 0})^2 - (dx^{\not 1})^2 - (dx^{\not 2})^2 - (dx^{\not 3})^2 \tag{11}$$

$$= (dx_{\not 0})^2 - (dx_{\not 1})^2 - (dx_{\not 2})^2 - (dx_{\not 3})^2$$

In order that the derivative has the appropriate properties with respect to the metric the derivative must also be coupled to the gamma matrices; a form appropriate for a quantum mechanical operator formalism is used; $\partial_\mu \rightarrow \frac{-i}{2}\partial_\mu\gamma_\mu \equiv \partial_{\not\mu}$; (note that this symbol is not the same as $\not\partial = \partial_\mu\gamma^\mu$ as in the former there is no sum over μ; the summation convention will be followed only for summation between one 'up-stairs' and one 'down-stairs' index; two identical indices either both 'up-stairs' or both 'down-stairs' are not summed). Note that $\partial^\dagger_{\not\mu} = -\partial^{\not\mu}$ so that $\overline{\partial_{\not\mu}} = -\partial_{\not\mu}$. The metric also has the following property;

$$g_{\not\alpha\not\epsilon} g^{\not\epsilon}{}_{\not\beta} = I_4\eta_{\alpha\beta} \tag{12}$$

that is, that in some sense the metric is the square root of the normal diagonal Lörentz metric. The inverse is simply;

$$g_{\not\alpha\not\epsilon} g^{\not\epsilon\not\beta} = I_4\eta_\alpha{}^\beta = \delta_{\not\alpha}{}^{\not\beta} \tag{13}$$

A group with a positive definite metric is compact (here it has an invariant length S equivalent to O_4). The unusual feature of metric (3) is that mathematical structures associated with the orthochronous Lörentz group, in this case the fundamental tensor and the spinor rep, are summed to generate a compact group.

But this results only obliquely by reference to a translated co-ordinate system; in the co-ordinate system appropriate for the metric, an anti-commuting co-ordinate system, the topology is non-compact!

III. COVARIANT DERIVATIVE AND VANISHING NON-METRICITY.

The next step is to generalize the metric (3) to non-inertial frames. In order to define covariant differentiation in anti-commuting co-ordinates special emphasis is placed on the vanishing of the covariant derivative of the metric; this is the mathematical expression of the principle of equivalence since it means that a co-ordinate system can be found in which the force in question - in G.R. this is gravitation but in the theory being developed here it will include electro-magnetism - can be 'transformed away'; a 'free-fall' frame exists *locally* in which the force is 'absent'. (The electro-magnetic field cannot be 'transformed away' by a Lörentz transformation in commuting co-ordinates).

Firstly form the generalized connection;

$$\Gamma^{\not\rho}_{\not\alpha\not\beta} = \frac{1}{2}\left(g_{\not\epsilon\not\alpha,\not\beta} + g_{\not\beta\not\epsilon,\not\alpha} - g_{\not\alpha\not\beta,\not\epsilon}\right)\bar{g}^{\not\epsilon\not\rho} \tag{14}$$

where the order of indices in eq.(14) is to be maintained rigorously. The dual is;

$$\bar{\Gamma}^{\not\rho}_{\not\alpha\not\beta} = \frac{1}{2}\bar{g}^{\not\rho\not\epsilon}\left(\overline{g_{\not\epsilon\not\alpha\,,\,\not\beta}} + \overline{g_{\not\beta\not\epsilon,\,\not\alpha}} - \overline{g_{\not\alpha\not\beta\,,\,\not\epsilon}}\right) \tag{15}$$

and again the order of indices is to be maintained. Note that the dual involves the derivative. The derivative $,\not\gamma$ is taken to be acting to the left. The bar operation is equal to hermitian conjugation followed by lowering (or raising) all indices;

$$(\overline{g_{\not\alpha\not\beta\,,\,\not\gamma}}) = (g_{\not\alpha^\dagger\not\beta^\dagger}\;\overleftarrow{\partial}_{\not\gamma^\dagger})^\dagger = -\,\overrightarrow{\partial}_{\not\gamma}\,\bar{g}_{\not\alpha\not\beta} = -\,\overrightarrow{\partial}_{\not\gamma}\,g_{\not\beta\not\alpha} \tag{16}$$

The definition of the dual connection is not required to define the covariant derivative of the metric but is needed to prove theorems for more general forms involving derivatives of vector fields. As we shall see, we are interested only in the situation that the derivative of the symmetric part of the metric vanishes. In that circumstance;

$$g_{\not\alpha\not\beta\,;\,\not\gamma} = \bar{g}_{\not\beta\not\alpha\,;\,\not\gamma} = -g_{\not\beta\not\alpha\,;\,\not\gamma}$$

and the order of indices in the differentiation is important since they do not necessarily commute. In keeping with the form of the notation the covariant derivative index is understood to be acting on the index to its immediate left. Thus the covariant derivative of the metric is written as;

$$g_{\not\alpha\not\beta\,;\,\not\gamma} = g_{\not\alpha(\not\beta\,;\,\not\gamma)} + \bar{g}_{\not\beta(\not\alpha\,;\,\not\gamma)} = g_{\not\alpha(\not\beta\,,\,\not\gamma)} + \bar{g}_{\not\beta(\not\alpha\,,\,\not\gamma)} - \Gamma^{\not\rho}_{\not\beta\not\gamma}\,g_{\not\alpha\not\rho} - \bar{\Gamma}^{\not\rho}_{\not\alpha\not\gamma}\,\bar{g}_{\not\beta\not\rho} \tag{17}$$

or alternatively, in the circumstance that the derivative of the symmetric part of the metric vanishes (which is the situation under consideration here) the connection is anti-symmetric in its lower two indices and we have;

$$\begin{aligned} g_{\not\alpha\not\beta\,;\,\not\gamma} = g_{\not\alpha(\not\beta\,;\,\not\gamma)} - g_{\not\beta(\not\alpha\,;\,\not\gamma)} &= g_{\not\alpha(\not\beta\,,\,\not\gamma)} - g_{\not\beta(\not\alpha\,,\,\not\gamma)} - \Gamma^{\not\rho}_{\not\beta\not\gamma}\,g_{\not\alpha\not\rho} + \Gamma^{\not\rho}_{\not\alpha\not\gamma}\,g_{\not\beta\not\rho} \\ &= g_{\not\alpha\not\beta\,,\,\not\gamma} - \Gamma^{\not\rho}_{\not\beta\not\gamma}\,g_{\not\alpha\not\rho} - \Gamma^{\not\rho}_{\not\gamma\not\alpha}\,\bar{g}_{\not\rho\not\beta} \\ &= 0 \end{aligned} \tag{18}$$

where the last line follows directly from the definition of the connection and the inverse property of the metric. Using (17) and (18) we have;

$$\bar{\Gamma}^{\not\phi}_{\not\alpha\not\gamma}\; g_{\not\phi\not\beta} = \Gamma^{\not\phi}_{\not\gamma\not\alpha}\; \bar{g}_{\not\phi\not\beta}$$

which is a relation derived from a general covariant derivative and should therefore hold for any rank two tensor in the theory if the covariant derivative of the metric is to vanish; i.e.,

$$\bar{\Gamma}^{\not\phi}_{\not\alpha\not\gamma}\; u_{\not\phi\not\beta} = \Gamma^{\not\phi}_{\not\gamma\not\alpha}\; \overline{u_{\not\phi\not\beta}} \tag{19}$$

for any rank two tensor u. This relation can be used to generalize the covariant derivative to higher rank tensors. In particular if the vector field is real ($\overline{A_{\not\mu}} = A_{\not\mu}$) and satisfies the Lörentz condition we have;

$$A_{\not\mu\,,\not\nu} = -g^{s}_{\not\mu\not\nu}A^{\not\phi}{}_{,\not\phi} - \overset{\rightharpoonup}{\partial}_{\not\nu}A_{\not\mu} = -\overset{\rightharpoonup}{\partial}_{\not\nu}A_{\not\mu} = \overline{A_{\not\mu\,,\not\nu}} \tag{20}$$

and by the definition of the bar operation for a purely real field;

$$A_{\not\mu\,;\not\nu} = A_{\not\mu\,,\not\nu} - \Gamma^{\not\phi}_{\not\mu\not\nu}A_{\not\phi} = \overline{A_{\not\mu\,,\not\nu}} - \bar{\Gamma}^{\not\phi}_{\not\nu\not\mu}\overline{A_{\not\phi}} = \overline{A_{\not\mu\,;\not\nu}} \tag{21}$$

Notice that the bar operation here involves the derivative and is the source of the odd permutation of un-contracted indices in the barred connection on the R.H.S. of eq(21). We can now define the general iterated covariant derivative for a real vector field $A_{\not\alpha}$ satisfying the Lörentz gauge (viz. eqs.(17), (19) and (21));

$$A_{\not\alpha\,;\,\not\beta\,;\,\not\gamma} = A_{\not\alpha\,;\,\not\beta\,,\,\not\gamma} - \Gamma^{\not\phi}_{\not\gamma\not\alpha}A_{\not\phi\,;\,\not\beta} - \Gamma^{\not\phi}_{\not\beta\not\gamma}A_{\not\alpha\,;\,\not\phi} \tag{22}$$

Note that in forming the covariant derivative an even permutation of the indices $\not\alpha$ $\not\beta$ and $\not\gamma$ is maintained on the R.H.S. of eq.(18) with respect to the order they appear in the metric on the L.H.S. (either g or $\bar{g}$ as the case may be) as we would do for spinor indices.

IV. CURVATURE AND COVARIANT DIFFERENTIATION IN ANTI-COMMUTING CO-ORDINATES

In the circumstance that the derivatives of the symmetric part of the metric vanish (gravity ignored; a free-fall frame) the connection is purely anti-symmetric in its lower indices (this nominally indicates a torsion tensor but in the present case, because the co-ordinates anti-commute, the physics represented will not in fact be torsion). In that context consider the *sum* of the following two-fold covariant derivatives of a real vector field; noting that in re-ordering the indices when expanding out in terms of connections an even permutation of un-contracted indices is always maintained when using the unbarred connection so that, for example, an index order $\not\alpha\ \not\beta\ \not\gamma$ can be rearranged as $\not\gamma\ \not\alpha\ \not\beta$ but not as $\not\alpha\ \not\gamma\ \not\beta$ which has an odd number of indices interchanged. The insertion of a dummy index does not change this since it still requires an odd number of interchanges to go from; $\not\alpha\ \not\beta\ \not\phi\ \not\gamma$ to $\not\alpha\ \not\gamma\ \not\phi\ \not\beta$. We have;

$$u_{\not\alpha\,;\,\not\beta\,;\,\not\gamma} + u_{\not\alpha\,;\,\not\gamma\,;\,\not\beta} \tag{23}$$

$$\begin{aligned}
&= u_{\not\alpha\,;\,\not\beta\,,\,\not\gamma} - \Gamma^{\not\phi}_{\not\gamma\not\alpha} u_{\not\phi\,;\,\not\beta} - \Gamma^{\not\phi}_{\not\beta\not\gamma} u_{\not\alpha\,;\,\not\phi} + u_{\not\alpha\,;\,\not\gamma\,,\,\not\beta} - \Gamma^{\not\phi}_{\not\beta\not\alpha} u_{\not\phi\,;\,\not\gamma} - \Gamma^{\not\phi}_{\not\gamma\not\beta} u_{\not\alpha\,;\,\not\phi} \\
&= u_{\not\alpha\,,\,\not\beta\,,\,\not\gamma} + \Gamma^{\not\phi}_{\not\alpha\not\beta\,,\not\gamma} u_{\not\phi} - \Gamma^{\not\phi}_{\not\alpha\not\beta} u_{\not\phi\,,\,\not\gamma} - \Gamma^{\not\phi}_{\not\gamma\not\alpha} u_{\not\phi\,,\,\not\beta} + \Gamma^{\not\phi}_{\not\gamma\not\alpha}\Gamma^{\not\rho}_{\not\phi\not\beta} u_{\not\rho} - \Gamma^{\not\phi}_{\not\beta\not\gamma} u_{\not\alpha\,,\,\not\phi} + \Gamma^{\not\phi}_{\not\beta\not\gamma}\Gamma^{\not\rho}_{\not\alpha\not\phi} u_{\not\rho} \\
&\quad + u_{\not\alpha\,,\,\not\gamma\,,\,\not\beta} + \Gamma^{\not\phi}_{\not\alpha\not\gamma\,,\not\beta} u_{\not\phi} - \Gamma^{\not\phi}_{\not\alpha\not\gamma} u_{\not\phi\,,\,\not\beta} - \Gamma^{\not\phi}_{\not\beta\not\alpha} u_{\not\phi\,,\,\not\gamma} + \Gamma^{\not\phi}_{\not\beta\not\alpha}\Gamma^{\not\rho}_{\not\phi\not\gamma} u_{\not\rho} - \Gamma^{\not\phi}_{\not\gamma\not\beta} u_{\not\alpha\,,\,\not\phi} + \Gamma^{\not\phi}_{\not\gamma\not\beta}\Gamma^{\not\rho}_{\not\alpha\not\phi} u_{\not\rho} \\
&= (\Gamma^{\not\phi}_{\not\alpha\not\beta\,,\not\gamma} + \Gamma^{\not\phi}_{\not\alpha\not\gamma\,,\not\beta} + \Gamma^{\not\rho}_{\not\gamma\not\alpha}\Gamma^{\not\phi}_{\not\rho\not\beta} + \Gamma^{\not\rho}_{\not\beta\not\alpha}\Gamma^{\not\phi}_{\not\rho\not\gamma}) u_{\not\phi} \\
&= \mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma} u_{\not\epsilon}
\end{aligned} \tag{24}$$

where, to obtain the second but last line in eq.(24) I have assumed that the vector field is massless and employed Proca's equation viz;

$$u_{\not\alpha\,,\,\not\beta\,,\,\not\gamma} + u_{\not\alpha\,,\,\not\gamma\,,\,\not\beta} = 2I_4\eta_{\beta\gamma}\partial_\epsilon\partial^\epsilon u_{\not\alpha} = 0 \tag{25}$$

and the antisymmetry of the 'slashed-gamma' $\Gamma^{\not\phi}_{\not\beta\not\alpha} = -\Gamma^{\not\phi}_{\not\alpha\not\beta}$ to define the 'curly-R' curvature tensor $\mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma}$. This form of the curvature tensor is to be compared with the conventional form; the latter is found by substituting a commutator for an anticommutator of the covariant derivatives in eq.(24) viz;

$$u_{\alpha\,;\,\beta\,;\,\gamma} - u_{\alpha\,;\,\gamma\,;\,\beta} = R^{\epsilon}_{\alpha\beta\gamma}u_\epsilon.$$

The 'curly-R' tensor has many properties similar to the conventional tensor and those of importance are proven in the appendix.

Using the Bianci identity and the symmetry of the contracted 'curly-hat' $\hat{\mathcal{R}}$ in its two un-contracted indices and which contains only products of first derivatives of the metric we have the following;

$$\begin{aligned}
0 &= g^{\not\beta\not\alpha}\hat{\mathcal{R}}_{\not\alpha\not\beta\,;\,\not\gamma} + g^{\not\beta\not\alpha}\hat{\mathcal{R}}_{\not\beta\not\gamma\,;\,\not\alpha} + g^{\not\beta\not\alpha}\hat{\mathcal{R}}_{\not\gamma\not\alpha\,;\,\not\beta} \\
&= g_s^{\not\beta\not\alpha}\hat{\mathcal{R}}_{\not\alpha\not\beta\,;\,\not\gamma} + g^{\not\beta\not\alpha}\hat{\mathcal{R}}_{\not\beta\not\gamma\,;\,\not\alpha} + g^{\not\alpha\not\beta}\hat{\mathcal{R}}_{\not\gamma\not\beta\,;\,\not\alpha} \\
&= \frac{-1}{2}\hat{\mathcal{R}}_{\,;\,\not\gamma} + 2g_s^{\not\beta\not\alpha}\hat{\mathcal{R}}_{\not\beta\not\gamma\,;\,\not\alpha} \\
&= \frac{-1}{2}\hat{\mathcal{R}}_{\,;\,\not\gamma} - \hat{\mathcal{R}}^{\not\alpha}{}_{\not\gamma\,;\,\not\alpha}
\end{aligned} \tag{26}$$

which allows us to write an 'Einstein-like' equation;

$$(\frac{1}{2}g^{\not\alpha\not\gamma}\hat{\mathcal{R}} + \hat{\mathcal{R}}^{\not\alpha\not\gamma})_{\,;\,\not\gamma} = 0 \tag{27}$$

Unlike the actual Einstein equation however the bracketed part of eq.(27) is traceless as follows. Lowering the $\not\alpha$ index and tracing;

$$\frac{1}{2}g_{\not\gamma}{}^{\not\gamma}\hat{\mathcal{R}} + \hat{\mathcal{R}}_{\not\gamma}{}^{\not\gamma} = -\hat{\mathcal{R}} + \hat{\mathcal{R}} = 0$$

because $g_{\not\gamma}{}^{\not\gamma} = -\frac{I_4}{2}\eta_{\not\gamma}{}^{\not\gamma} = -2I_4$. We wish to equate eq.(27) to the vanishing divergence of a stress-energy tensor $T^{\not\alpha\not\gamma}{}_{;\not\gamma} = 0$. It is clear that this equation must be satisfied independently for any symmetric and anti-symmetric components of the stress-energy tensor (we may reasonably assume the anti-symmetric part of the stress-energy tensor is zero) and thus equating symmetric parts immediately leads to;

$$(+\frac{1}{2}g_s^{\not\alpha\not\gamma}\hat{\mathcal{R}} + \hat{\mathcal{R}}^{\not\alpha\not\gamma})_{\,;\,\not\gamma} = 0 = T_s^{\not\alpha\not\gamma}{}_{\,;\,\not\gamma}$$

Replacing the metric $g_s^{\not\alpha\not\gamma}$ by its constant symmetric part $\frac{-I_4}{2}\eta^{\alpha\gamma}$, and noting that because $\mathcal{R}^{\not\alpha\not\gamma}$ is symmetric we may drop the slash notation on the symmetric tensors and multiply through by the constant $\frac{1}{2}\gamma^{\gamma}$ to eliminate the slash on the derivative index to obtain;

$$I_4(-\frac{1}{4}\eta^{\alpha\gamma}\hat{\mathcal{R}} + \hat{\mathcal{R}}^{\alpha\gamma})_{\,;\,\gamma} = 0 = cI_4T^{\alpha\gamma}{}_{\,;\,\gamma} \tag{28}$$

which we recognize, dropping the I_4's, as the correct form for the stress-energy equation for a free massless field. (Of course the L.H.S. of eq.(28) - the curvature with un-slashed co-ordinates - is only meaningful if referred to a 'spinorial' observer observing in anti-commuting co-ordinates). The constant c must have dimension l^2 because the stress-energy tensor has the dimension l^{-4} and the curvature tensor has the dimension l^{-2}. Thus the scalar curvature, and Lagrangian, is given by;

$$k\,\hat{\mathcal{R}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = +\frac{1}{4}F_{\mu\nu}F^{\nu\mu} \tag{29}$$

where the constant k is of dimension of inverse length squared. This constant will not be related to the gravitational constant as such but must be a function of the

length scale of the source over which the charge is distributed. As a reasonable ansatz we may take;

$$k \propto \sqrt{|g|}\,\alpha^2 m_0^2 \tag{30}$$

where α is the fine-structure constant, $|g|$ is the modulus of the determinant of the metric and m_0 is the rest mass of the source.

V. POTENTIAL AND SOURCE TERMS

If the speculation entered into at the beginning of this paper is valid we expect to find curved anti-commuting co-ordinates manifestly in the vicinity of charged particles such as an electron or proton. Imagine, with at least some degree of creative license, that you are an observer sitting on an electron. What does the space-time in your vicinity look like? The thing of note is that, if you rotate by 2π around a fixed axis, your co-ordinates anti-commute. Your space-time is not the same as expected for a macroscopic 'bosonic' observer. Thus we actually expect the anti-commuting space-time to dominate in the vicinity of a fermion; indeed, the fermion itself must be actually generating this aspect of local space-time - it must be the source. (Analogously we must expect the gravitating masses of the universe to be, not just the source of the gravitational field, but the source of commuting space-time itself if this scenario is valid).

We want to interpret this anti-commuting aspect of space-time, superimposed upon gravity-generated commuting space-time, as the electric field of the charged particle. We can encompass two kinds of charges with anti-commuting space-

time because, unlike with the case of the usual commuting co-ordinates, complex-conjugate types of space-time in anti-commuting co-ordinates can be defined; $x^{\not\alpha}$ and $x^{\dagger\not\alpha} \equiv \frac{1}{2}x^{\alpha}\gamma^{\dagger\alpha}$ (no sum on α). The theory is symmetric between these two so that, for example, we have the hermitian conjugate metric coupling to the hermitian conjugate co-ordinates;

$$\bar{g}^{\dagger}_{\not\alpha\not\beta}x^{\dagger\not\beta} = x_{\not\beta}\bar{g}^{\not\beta\not\alpha} = x^{\not\alpha} = x^{\dagger}_{\not\alpha}$$

and so-on for the other relations given in the appendix. Since $x^{\alpha}\neq x^{\dagger\alpha}$ there are in fact two possible types of anti-commuting space-times which we will associate with the two possible charge types that exist in the universe; positive and negative. But note that, in the given representation, they do not differ in their time representation since $\gamma^0 = \gamma^{\dagger 0}$.

The question naturally arises as to which type of space-time, commuting or anti-commuting, dominates in the quantum domain. Dominates is perhaps not quite the correct term here as the issue is the ‘stiffness’ of the space-time. Because of the ‘stretching’ of the commuting space-time with respect to anti-commuting space-time with expansion of the universe we expect the coupling constants of the two forces to be related as a function of the degree of ‘stretching’ of space-time. Thus commuting space-time is more than 30 orders of magnitude ‘stiffer’ than anti-commuting space-time. Thus in the quantum domain in the current epoch we expect the anti-commuting space-time to be much more ‘flexible’ than the commuting space-time. Only in the very early history of the universe when the coupling constants of gravitation and electro-magnetism are more equal does the commuting space-

time begin to be in parity with the anti-commuting space-time. But note that the cancelation between different charged space-times only occurs for the space dimensions;

$$\gamma^i + \gamma^{\dagger i} = \gamma^i - \gamma^i = 0 \ ; \ \ \gamma^0 + \gamma^{\dagger 0} = 2\gamma^0 \tag{31}$$

and the anti-commuting time-part of space-time will expand with the commuting space-time and the observed macroscopic time dimension is then the sum of two parts; one generated by the commuting part of space-time and one generated by the anti-commuting part.

Of course this also means that the time dimension in anti-commuting space-time in the vicinity of a charged particle is 'stiff'; because it is stretched like the commuting space-time it does not bend easily - only the space dimensions are significantly bent in the region of a charged particle and the time is effectively 'flat'. Thus the symmetry of the charged object with respect to its electro-magnetic field may be represented, to a good approximation, by a compact space. Is it perhaps for this reason that spinorial particles are represented by a compact representation of the Lörentz group; SU(2) is in fact a representation of SO(3) up to a sign? Note that the stiffness of the time dimension does not effect the description of free-fields since the photon has no longitudinal or time-like polarization (any electro-magnetic bending of the 'stiff' time dimension is suppressed by more than 30 orders of magnitude).

Putting aside the issue of the dual nature of the time dimension in this theory we may ask how might it be possible to incorporate the basic features of quantum physics into the metric structure to describe the source? The non-vanishing $x^{\emptyset}$

component of the vacuum remote from charged sources would not appear to effect the free-field tensor since the longitudinal polarization of a photon is zero and the photon does not cause oscillations in the 'stiff' time dimension (or at least these are suppressed to the order of the gravitational coupling constant). However, we may modify the metric to represent the anti-commuting space-time in the vicinity of the source by neglecting the time co-ordinate and treating the space as compact. Consider the following metric;

$$g_{\not i \not j} = \frac{1}{2} e^{i\frac{2\pi}{\hbar} p\cdot x} (\delta_{\not i \not j} + i\sigma_{\not i \not j}) = \bar{g}_{\not j \not i} \tag{32}$$

where $p{\cdot}x = p_{\not\epsilon} x^{\not\epsilon} = p_\epsilon x^\epsilon$ and p_ϵ is the particle four-momentum. (The index raising form is made by setting p=0). Now the wave factor introduced into the metric means that the distance $ds^2 = Tr.dx^{\not\alpha} \bar{g}_{\not\alpha\not\beta} dx^{\not\beta}$ oscillates in value between positive and negative values in this case of *spatial distance* since, although the wave factor contains a $p_0 x^0$ term, it is part of a scalar which may be referred to the commuting co-ordinates as above. We assume that such a metric is valid in the immediate vicinity of a source. Because the metric is complex and referred to complex co-ordinates ds^2 never vanishes in spite of the phase-factor. By the principle of relativity this should also be true for the observer with respect to commuting co-ordinates. If the imaginary part of the metric is unmeasurable (that is, if the real part only is measurable) then the imaginary part may be represented by the uncertainty of the measurement. Now the phase factor oscillates with both real and imaginary components and the metric (32) contains a real part $\eta_{i\,j}$ and a pure imaginary part $i\sigma_{i\,j}$. Now;

$$|i\sigma_{i\,j}|^2 = 6 \text{ and } |\eta_{i\,j}|^2 = 3$$

so that the imaginary part of metric (32) is maximised when the phase factor is purely real. Thus the imaginary part of the metric is maximised for the corresponding angle in the phase factor;

$$\frac{2\pi n}{\hbar} p{\cdot}x = 2\pi n \tag{33}$$

where n is a constant integer (we exclude the trivial case where $n = 0$). The measurement of the real part of the corresponding product will thus have an equal or greater uncertainty in its measurement than the value specified by eq.(33). We thus obtain;

$$|\triangle p{\cdot}\triangle x| \geq \hbar \tag{34}$$

Thus the presence of a source phase-factor representing a wave-function in the metric related to the anti-commuting part of space-time may be interpreted as an expression of the uncertainty principle when referred to commuting co-ordinates (i.e. the observers co-ordinate system). That the uncertainty only applies in the quantum scale of things results from the effective vanishing of anti-commuting space-time over macroscopic distances.

Armed with this idea can we extract an expression for a charged source tensor from metric (32)? For starters we note that the given derivation of the 'curly-$\hat{\mathcal{R}}$' curvature tensor breaks-down; it is predicated on an object satisfying a massless Proca's equation and which also satisfies the Lörentz condition (although the latter is actually imposed by the mathematical constraints of the theory; see appendix).

That is, the derivation of the free-field curvature tensor has relied on a massless vector field whilst the sources are massive spinors. It is also predicated on the vanishing of the derivative of the symmetric part of the metric; and this condition breaks down with the insertion of a phase factor in metric (32). To study spinors the mathematical machinery must be re-geared and this has not yet been successfully achieved.

We might nevertheless attempt an ansatz based on general ideas. The key difference between the 'Einstein-like' conservation equation resulting from the contraction of the Bianci identity for the 'curly-R' curvature tensor and that for General Relativity is the tracelessness of the former. In the presence of a source the trace of the equation will not vanish and we require an equation of the form (for convenience dropping the slash notation under the assumption that the equation is understood to imply curvature of the anti-commuting part of space-time);

$$k(-\frac{1}{4}\eta^{\alpha\beta}\hat{\mathcal{R}}_{FF} + \hat{\mathcal{R}}^{\alpha\beta}_{FF}) + k'\hat{\mathcal{R}}^{\alpha\beta}_{S} = T^{\alpha\beta} \tag{35}$$

where $\hat{\mathcal{R}}^{\alpha\beta}_{S}$ is a curly-R type curvature tensor for anti-commuting co-ordinates and contains analogous components from the contraction of a Bianci identity related to a source metric with non-vanishing derivatives (it may contain a scalar piece multiplied by the metric for example). The new constant k' here is, of course, of dimension l^{-2}.

Of course to construct $\hat{\mathcal{R}}_S$ we must replace the massless vector field in eq.(25) by a massive spinor field in the derivation of the curvature tensor;

$$u_{\not\alpha\,,\,\not\beta\,,\,\not\gamma} + u_{\not\alpha\,,\,\not\gamma\,,\,\not\beta} = 2I_4\eta_{\beta\gamma}\partial_\epsilon\partial^\epsilon u_{\not\alpha} = 2g^{s}_{\not\beta\not\gamma}m_0^2 u_{\not\alpha} \tag{36}$$

so that a constant multiple of the mass squared is added to the definition of the curvature tensor derived from the anti-commutator of eq.(24). One might guess from this that the appropriate addition to the stress-energy tensor for the free-field would then be a constant multiple of the mass-squared and for the vector piece the product of momenta $p_{\not\alpha}p_{\not\beta}$ which, with dropping of the slash notation and appropriate adjustment of constants, can be converted into the conventional source stress-energy tensor for a charged particle [20].

Because of the symmetry of the contracted curly-R curvature tensor in its two un-contracted indices for this to be a viable approach we will require that the momenta commute so that this can be re-written as $2a.p_{\not\beta}p_{\not\gamma}$; however one expects the momenta in anti-commuting co-ordinates to anti-commute so that this tensor reduces to a trivial result. An alternative may be to consider the analogue of conventional curvature in anti-commuting space-time to feed into a Bianci identity to construct a conserved quantity;

$$\check{\mathcal{R}}_{s\not\gamma\not\beta\not\alpha}{}^{\not\alpha} = \Gamma^{\not\epsilon}_{\not\beta\not\gamma}\Gamma^{\not\alpha}_{\not\epsilon\not\alpha} + \Gamma^{\not\epsilon}_{\not\alpha\not\gamma}\Gamma^{\not\alpha}_{\not\epsilon\not\beta} \tag{37}$$

where once again I have dropped the terms involving second derivatives as they will vanish. Note that for metric eq.(32) the derivative of the symmetric part of the metric does not vanish and the connection is not completely anti-symmetric. The tensor $\check{\mathcal{R}}_{s\not\gamma\not\beta\not\alpha}{}^{\not\alpha}$ will thus be expected to contain both a symmetric and an anti-symmetric part or more generally be expressible as the sum of two tensors; one symmetric and one anti-symmetric. We may expect, with regard to the source momenta, that the symmetric component will be trivial and so the conserved quantity will be then expressed

as an anti-symmetric tensor which must be coupled to a stress-energy tensor which is also anti-symmetric. Conventional wisdom dictates that such a stress-energy tensor will not conserve angular momentum but this may not be the case here for spinors in anti-commuting space-time. The resolution of these issues, if indeed they can be resolved, must await development of a means of adapting the derivation of curly-R curvature tensors from spinor fields; which as mentioned is work still outstanding.

VI. CONCLUSION

The reader at this stage may well object, with some justification, "what has been achieved by this construction since we have a good theory of electro-magnetic processes in the form of Q.E.D. to which the above theory adds no new prediction and cannot, at least at its current level of development, match current theory?" It is difficult to disagree with this complaint except to add that at some point one must try to incorporate gravity into the scheme of things since its omission is clearly a physical impossibility. This new approach however does hold some promise of new physical predictions at extreme energies where the coupling constants of gravitation and electro-magnetism are comparable. Also the ability to give an account of the wide difference in the coupling constants of the two forces at low energy is a useful feature and somthing unaccounted for in standard theory (Kaluza-Klein theory not withstanding). The ability to give an account for the origin of the uncertainty principle on the basis of pure geometry seems to be a new feature in physics as far as I can ascertain.

Attempts at geometric unification are of course nothing new dating back to the

work of Nördstrom, Weyl, Kaluza-Klein, Eddington, Cartan and Einstein. (For a review of the history of the subject see [4]; for more recent work in a similar vein to the Weyl-Eddington approach see [9]. Standard works involving asymmetric tensor fields can be found in Einstein [6], Kibble and Sciami [10], [13], [14]. For more recent extensions to Yang-Mills fields see [15], [16], [17] and [18] and for an extensive recent review of the subject see [12]. Note however that something rather different is being developed in this current paper to most previous attempts; the coupling of the anti-symmetric connection to the anti-symmetric co-ordinate system means that the connection does not represent torsion).

This work should then be seen in this light; Q.E.D. clearly contains profound truth - its empirical validity tells us this - but neither it nor its electro-weak (standard model) extension can be complete. To couple it to gravitation we need something completely new. The present work follows on from many previous attempts but avoids the problems associated with introduction of a torsion field; for anti-commuting co-ordinates the anti-symmetric connection propagates;

$$\Gamma^{\not\alpha}_{[\not\beta\not\gamma]}\frac{dx^{\not\beta}}{ds}\frac{dx^{\not\gamma}}{ds}\neq 0 \tag{38}$$

unlike the case for commuting co-ordinates. (In which case it is interpreted as a spin-contact interaction - see [7] and [8]). Mixing of commuting and anti-commuting space-time results in a new symmetry. This new symmetry is encapsulated in the expression of both gravitation and electro-magnetism in terms of curvature of space-time; in the case of gravitation the curvature is of the commuting part of space-time and in the case of electro-magnetism of the anti-commuting part. The expression of

both gravitation and electro-magnetism in terms of general co-ordinate transformations implies invariance of the laws of nature with respect to arbitrary mixtures of commuting and anti-commuting co-ordinates although no formal proof of this statement has yet been developed; the laws of the universe should be valid for general co-ordinate transformations in the current epoch and at all preceding and succeeding times as the mix will vary with the age of the universe if the suppositions presented in this paper are correct.

On the other hand one may ask what happens to particle statistics when, as in the early universe, commuting space-time is as flexible as anti-commuting space-time? Is the electron a spinor precisely because its major physical manifestation is the generation and bending of adjacent anti-commuting space-time? And if this is the case shouldn't it manifest bosonic characteristics if the commuting space-time it is generating reaches equal flexibility with the anti-commuting space-time near the Planck scale?

It is possible that these conclusions are a consequence of the theory; that is, that objects such as electrons manifest continuous statistics which only becomes apparent at very high energy. However it is also possible that the continuous statistics require quantization; by which is meant that, instead of continuous statistics for an object like an electron, one encounters a new particle at high energy which is a bosonic super-partner of the electron that 'carries off' the 'bosonic' part of the electron statistics at high energy. In that case there would be a more conventional type of supersymmetry (if indeed supersymmetry may be called conventional).

A comment on the issue of the principle of equivalence is appropriate as it is so

much at the center of General Relativity. It is represented by the vanishing of the covariant derivative of the metric and this, in turn, tells us that a frame exists in anti-commuting space-time in which the electro-magnetic field can be 'transformed away' *locally*; the analogue of a free-fall frame in commuting space-time. This in turn means that the physics can be described by curvature in the anti-commuting co-ordinates.

But what are we to make of the idea of the equivalence of charge and mass expected near the Planck scale? We know that charge comes in fixed quantum integral multiples of $\frac{1}{3}$ but particle masses vary considerably for the same unit of charge. Both mass and charge vary with energy but not in the same way; the coupling constant of the E.M. field $\alpha \approx 137^{-1}$ varies with energy but not in a way that is easily related to a Lörentz boost by a multiplicative constant (it takes the value $\approx 128^{-1}$ at the weak unification scale and is thought to take a value order unity at the Planck scale). Clearly the symmetry between mass and charge is badly broken in nature. If the central surmise of this paper is correct the symmetry would be exact only in the situation that the universe consisted solely of a single electron with photons, and gravitons ($\pm$ their corresponding supersymmetry partners?). The universe is however much more complex than this. The existence of the weak interaction means that the concept of charge is related to a more complex vacuum structure than that represented in this paper leading to massive electro-magnetically charged non-composite fundamental bosons. (The vacuum of the presented theory is represented by the effective vanishing of the $x^{\not{j}}$ anti-commuting co-ordinates in regions remote from charged sources; if these are non-vanishing the principle of

equivalence of charge and mass means that the vacuum energy is altered - it also means that the intrinsic statistics of the vacuum are altered. One intriguing possibility is that the rate of expansion of the $x^{\emptyset}$ time component is slightly retarded with respect to x^0 embedding a direction in the time dimension and modifying the vacuum structure). Moreover the Higgs field alters the masses of the particles as does the strong interaction - which introduces a further complication in terms of color charge. However, the most important physical feature which appears to break the symmetry between charge and mass is the stretching of the commuting space-time due to expansion of the universe and more specifically the observation that the anti-commuting time dimension stretches like the commuting time dimension leading to a profound disturbance of any simple symmetry between commuting and anti-commuting space-time and hence between mass and charge. The charge ends up related to a compact group and the mass to a non-compact one.

Lastly the deficiencies of the current approach need to be admitted. The mathematical structure is predicated on a massless spin-one field. Much work remains to extend the theory to massive spinors for successful incorporation of sources. Nevertheless there is enough scope in the theory to hope that a way may be found in the future. The theory may also have potential for extension to non-abelian Yang-Mills fields but at the moment this is lacking.

APPENDIX A: NOTATION AND CONVENTIONS

The co-ordinates are coupled to 4x4 gamma matrices and represented with a 'slash' notation;

$$x_{\not 0} \equiv x_0 \frac{\gamma_0}{2} \ , \ x_{\not 1} \equiv x_1 \frac{\gamma_1}{2} \ , \ x_{\not 2} \equiv x_2 \frac{\gamma_2}{2} \ , \ x_{\not 3} \equiv x_3 \frac{\gamma_3}{2}$$

Where the 4x4 gamma matrices satisfy;

$$[\gamma_\alpha, \gamma_\beta]_- = 2 I_4 \eta_{\alpha\beta} \text{ and } \eta_{\alpha\beta} = \text{diag.}(+,-,-,-)_{\alpha\beta}$$

Indices are raised or lowered by the 16x16 metric;

$$g_{\not\alpha\not\beta} = -\frac{1}{2}(I_4 \eta_{\alpha\beta} - i\sigma_{\alpha\beta}); \ , \ \ \bar{g}_{\not\alpha\not\beta} = -\frac{1}{2}(I_4 \eta_{\alpha\beta} + i\sigma_{\alpha\beta})$$

and hence $\bar{g}_{\not\alpha\not\beta} = g_{\not\beta\not\alpha}$, and we have;

$$\bar{g}_{\not\beta\not\alpha} x^{\not\alpha} = x_{\not\beta} \ , \ g_{\not\beta\not\alpha} x^{\not\alpha} = -2 x_{\not\beta} \ , \ g_{\not\beta\not\alpha} x^{\not\alpha\dagger} = x_{\not\beta} \ , \ \bar{g}_{\not\beta\not\alpha} x^{\not\alpha\dagger} = -2 x_{\not\beta}. \tag{A1}$$

Metric products have the following properties;

$$g_{\not\alpha\not\beta} g^{\not\beta\not\epsilon} = \delta_{\not\alpha}{}^{\not\epsilon} = \bar{g}_{\not\alpha\not\beta} \bar{g}^{\not\beta\not\epsilon} = I_4 \delta_\alpha{}^\epsilon$$

$$\bar{g}_{\not\alpha\not\beta} g^{\not\beta\not\epsilon} = g_{\not\alpha}{}^{\not\epsilon} \ \ ; \ \ g_{\not\alpha\not\beta} g^{\not\epsilon\not\beta} = g_{\not\alpha}{}^{\not\epsilon}$$

APPENDIX B: TRANSFORMATION PROPERTIES OF THE METRIC

Gamma matrices and sigma matrices transform under Lörentz transformations as follows;

$$S^{-1} \gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu \quad \text{ and } \quad S^{-1} \sigma^{\mu\nu} S = \Lambda^\mu{}_\lambda \Lambda^\nu{}_\kappa \sigma^{\lambda\kappa}$$

where $S \approx (1 + \frac{i}{2} \epsilon^{\mu\nu} \sigma_{\mu\nu})$ and $\Lambda^\nu{}_\nu \approx (g^\mu{}_\nu + \epsilon^\mu{}_\nu)$ with the infinitesimal $\epsilon^{\mu\nu}$ antisymmetric in μ and ν. Neither the gamma matrices nor the sigma matrices commute with the spinor representation of the Lörentz group S so that neither transforms as an

invariant tensor under the Lörentz group. However, the modification of the metric by the inclusion of an antisymmetric σ matrix term *when coupled to anti-commuting co-ordinates* does transform appropriately as we will now see. The generalization to gamma-slashed co-ordinates is (to first order in infinitesimal);

$$\Lambda^{\mu}{}_{\nu} \rightarrow \Lambda^{\not{\mu}}{}_{\not{\nu}} \approx (g^{\not{\mu}}{}_{\not{\nu}} + \epsilon^{\not{\mu}}{}_{\not{\nu}}) \tag{B1}$$

where g now includes both a symmetric and an antisymmetric part. A bar $\bar{\Lambda}$ is as per the metric bar operation and indicates hermitian conjugation plus raising or lowering of gamma-matrix coupled indices so that;

$$\overline{\epsilon_{\not{\alpha}}{}^{\not{\beta}}} = (\epsilon^{\not{\alpha}}{}_{\not{\beta}})^{\dagger} = \epsilon^{\not{\beta}}{}_{\not{\alpha}} = -\epsilon_{\not{\alpha}}{}^{\not{\beta}}.$$

The co-ordinates transform as (from properties A1);

$$x^{\not{\alpha}}\Lambda^{\not{\alpha}'}{}_{\not{\alpha}} = x^{\not{\alpha}}\bar{\Lambda}_{\not{\alpha}}{}^{\not{\alpha}'} = x^{\not{\alpha}'} \quad \text{and} \quad \bar{\Lambda}^{\not{\alpha}'}{}_{\not{\alpha}}x^{\not{\alpha}} = \Lambda_{\not{\alpha}}{}^{\not{\alpha}'}x^{\not{\alpha}} = x^{\not{\alpha}'}$$

Now defined as (to first order in the infinitesimals);

$$\begin{aligned}
\bar{\Lambda}_{\not{\alpha}'}^{\not{\alpha}} g_{\not{\alpha}\not{\beta}} \Lambda_{\not{\beta}'}^{\not{\beta}} &= (\bar{g}_{\not{\alpha}'}^{\not{\alpha}} + \epsilon_{\not{\alpha}'}^{\not{\alpha}}) g_{\not{\alpha}\not{\beta}} (g_{\not{\beta}'}^{\not{\beta}} + \epsilon_{\not{\beta}'}^{\not{\beta}}) \\
&= (\bar{g}_{\not{\alpha}'}^{\not{\alpha}} + \epsilon_{\not{\alpha}'}^{\not{\alpha}})(g_{\not{\alpha}\not{\beta}'} + 2\epsilon_{\not{\alpha}\not{\beta}'}) \\
&= g_{\not{\alpha}'\not{\beta}'} - 2\epsilon_{\not{\alpha}\not{\beta}'} + 2\epsilon_{\not{\alpha}\not{\beta}'} \\
&= g_{\not{\alpha}'\not{\beta}'}
\end{aligned} \tag{B2}$$

and similarly one can show that $\bar{\Lambda}_{\not{\alpha}'}^{\not{\alpha}} \bar{g}_{\not{\alpha}\not{\beta}} \Lambda_{\not{\beta}'}^{\not{\beta}} = \bar{g}_{\not{\alpha}'\not{\beta}'}$ so that we obtain the relations;

$$\begin{aligned}
ds^2 &= Tr.dx^{\not{\alpha}'}dx_{\not{\alpha}'} = Tr.dx^{\not{\alpha}}\Lambda^{\not{\alpha}'}{}_{\not{\alpha}}\bar{g}_{\not{\alpha}'\not{\beta}'}\bar{\Lambda}^{\not{\beta}'}{}_{\not{\beta}}dx^{\not{\beta}} = Tr.dx^{\not{\alpha}}\bar{g}_{\not{\alpha}\not{\beta}}dx^{\not{\beta}} = ds^2 \\
ds^2 &= (Tr.g)^{-1}Tr.dx^{\not{\alpha}}\bar{\Lambda}_{\not{\alpha}}{}^{\not{\alpha}'}g_{\not{\alpha}'\not{\beta}'}\Lambda_{\not{\beta}}{}^{\not{\beta}'}dx^{\not{\beta}} = (Tr.g)^{-1}Tr.dx^{\not{\alpha}}g_{\not{\alpha}\not{\beta}}dx^{\not{\beta}} = ds^2
\end{aligned} \tag{B3}$$

(Note; the inverse trace g in the last expression occurs because of metric contraction properties A1 and has the constant value $-\frac{1}{2}$). The metric thus acts as an invariant tensor under the *modified* co-ordinate system and ds^2 is an invariant.

APPENDIX C: PROOF OF THE LÖRENTZ IDENTITY

This applies to the free-field in which case we set the derivative of the symmetric part of the metric to zero i.e. $\eta_{\not{\alpha}\not{\beta}\,,\,\not{\gamma}} = 0 \;\; ; \; \{\forall \;\; \not{\gamma}\}$. Using the identities;

$$g_{\not{\alpha}\not{\phi}}\bar{g}^{\not{\phi}}{}_{\not{\beta}} = g_{\not{\alpha}\not{\beta}} \;\; \text{and} \;\; \bar{g}_{\not{\alpha}}{}^{\not{\phi}} g_{\not{\phi}\not{\beta}} = \bar{g}_{\not{\alpha}\not{\beta}} = g_{\not{\beta}\not{\alpha}}$$

we have; $g_{\not{\alpha}\not{\beta}\,,}{}^{\not{\alpha}} = g_{\not{\alpha}\not{\beta}\,,}{}^{\not{\phi}}\;\bar{g}_{\not{\phi}}{}^{\not{\alpha}} = -\frac{i}{2}\sigma_{\alpha\beta\,,}{}^{\not{\phi}}\;\bar{g}_{\not{\phi}}{}^{\not{\alpha}}$ and also $g_{\not{\alpha}\not{\beta}\,,}{}^{\not{\alpha}} = g_{\not{\alpha}}{}^{\not{\phi}}\bar{g}_{\not{\phi}\not{\beta}\,,}{}^{\not{\alpha}} = \bar{g}^{\not{\alpha}}{}_{\not{\phi}} g_{\not{\beta}\not{\alpha}\,,}{}^{\not{\phi}} = +\frac{i}{2}\bar{g}^{\not{\alpha}}{}_{\not{\phi}}\sigma_{\alpha\beta\,,}{}^{\not{\phi}}$ whence;

$$-\frac{i}{2}\sigma_{\alpha\beta\,,}{}^{\not{\phi}}\;\bar{g}_{\not{\phi}}{}^{\not{\alpha}} \;\; -\frac{i}{2}\bar{g}^{\not{\alpha}}{}_{\not{\phi}}\sigma_{\alpha\beta\,,}{}^{\not{\phi}} \;\; = 0$$

$$\text{or;} \qquad \sigma_{\alpha\beta\,,}{}^{\not{\phi}}\;(\bar{g}_{\not{\phi}}{}^{\not{\alpha}} + g_{\not{\phi}}{}^{\not{\alpha}}) = \sigma_{\alpha\beta\,,}{}^{\not{\phi}}\; I_4\eta_{\phi}{}^{\alpha} = \sigma_{\alpha\beta\,,}{}^{\not{\alpha}} \;\; = 0 \qquad \text{(C1)}$$

where the second line follows since, with vanishing of the derivative of any symmetric component generated in the commutation of the sigma matrices and the picking up of an extra minus sign from the commutation of the sigma matrix with the derivative index $\not{\partial}^{\phi}$ we get;

$$\begin{aligned} \sigma_{\alpha\beta\,,}{}^{\not{\phi}}\bar{g}_{\not{\phi}}{}^{\not{\alpha}} &= \sigma_{\alpha\beta\,,}{}^{\not{\phi}}\frac{1}{2}I_4(\eta_{\phi}{}^{\alpha} - i\sigma_{\phi}{}^{\alpha}) \qquad \text{(C2)} \\ &= \frac{1}{2}(I_4\eta_{\phi}{}^{\alpha}\sigma_{\alpha\beta\,,}{}^{\not{\phi}} - i\sigma_{\alpha\beta\,,}{}^{\not{\phi}}\sigma_{\phi}{}^{\alpha}) \\ &= \frac{1}{2}(I_4\eta_{\phi}{}^{\alpha}\sigma_{\alpha\beta\,,}{}^{\not{\phi}} - i\sigma_{\phi}{}^{\alpha}\sigma_{\alpha\beta\,,}{}^{\not{\phi}}) \\ &= \bar{g}_{\not{\phi}}{}^{\not{\alpha}}\sigma_{\alpha\beta\,,}{}^{\not{\phi}} \end{aligned}$$

Where the slash notation has been dropped when the metric is expanded out into its constituent symmetric and anti-symmetric parts. The Lörentz condition ensures that for any derivative of the free field component e.g. $g^{A}_{\not\alpha\not\beta\,,\,\not\gamma}$ the three indices must label different values. i.e. $\alpha \neq \beta \neq \gamma \neq \alpha$.

APPENDIX D: CURVATURE TENSOR IDENTITIES

In the case that the derivatives of the symmetric part of the metric vanish the connection is anti-symmetric in its lower two indices (a torsion connection in commuting co-ordinates but not in anti-commuting co-ordinates). The antisymmetry of the connection leads to the following identities.

The contracted curvature tensor is symmetric in its un-contracted indices;

$$\mathcal{R}^{\not\alpha}_{\not\alpha\not\beta\not\gamma} = \mathcal{R}_{\not\beta\not\gamma} = \mathcal{R}_{\not\gamma\not\beta} \tag{D1}$$

which follows directly from eq.(24).

The curvature tensor (and its barred analogue formed from the barred form of the connection) has the following cyclic property;

$$\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma} + \bar{\mathcal{R}}^{\not\epsilon}_{\not\beta\not\gamma\not\alpha} + \bar{\mathcal{R}}^{\not\epsilon}_{\not\gamma\not\alpha\not\beta} = \mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma} + \mathcal{R}^{\not\epsilon}_{\not\beta\not\gamma\not\alpha} + \mathcal{R}^{\not\epsilon}_{\not\gamma\not\alpha\not\beta} = 0 \tag{D2}$$

viz;

$$\begin{aligned}
\mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma} + \mathcal{R}^{\not\epsilon}_{\not\beta\not\gamma\not\alpha} + \mathcal{R}^{\not\epsilon}_{\not\gamma\not\alpha\not\beta} &= \Gamma^{\not\epsilon}_{\not\alpha\not\beta\,,\not\gamma} + \Gamma^{\not\epsilon}_{\not\alpha\not\gamma\,,\not\beta} + \Gamma^{\not\phi}_{\not\gamma\not\alpha}\Gamma^{\not\epsilon}_{\not\phi\not\beta} + \Gamma^{\not\phi}_{\not\beta\not\alpha}\Gamma^{\not\epsilon}_{\not\phi\not\gamma} \\
&\quad + \Gamma^{\not\epsilon}_{\not\beta\not\gamma,\not\alpha} + \Gamma^{\not\epsilon}_{\not\beta\not\alpha\,,\not\gamma} + \Gamma^{\not\phi}_{\not\alpha\not\beta}\Gamma^{\not\epsilon}_{\not\phi\not\gamma} + \Gamma^{\not\phi}_{\not\gamma\not\beta}\Gamma^{\not\epsilon}_{\not\phi\not\alpha} \\
&\quad + \Gamma^{\not\epsilon}_{\not\gamma\not\alpha\,,\not\beta} + \Gamma^{\not\epsilon}_{\not\gamma\not\beta\,,\not\alpha} + \Gamma^{\not\phi}_{\not\beta\not\gamma}\Gamma^{\not\epsilon}_{\not\phi\not\alpha} + \Gamma^{\not\phi}_{\not\alpha\not\gamma}\Gamma^{\not\epsilon}_{\not\phi\not\beta} \\
&= \Gamma^{\not\epsilon}_{\not\alpha\not\beta\,,\not\gamma} + \Gamma^{\not\epsilon}_{\not\alpha\not\gamma\,,\not\beta} + \Gamma^{\not\phi}_{\not\gamma\not\alpha}\Gamma^{\not\epsilon}_{\not\phi\not\beta} + \Gamma^{\not\phi}_{\not\beta\not\alpha}\Gamma^{\not\epsilon}_{\not\phi\not\gamma}
\end{aligned}$$

$$
\begin{aligned}
&-\Gamma^{\not\epsilon}_{\not\gamma\not\beta,\not\alpha}-\Gamma^{\not\epsilon}_{\not\alpha\not\beta\,,\not\gamma}-\Gamma^{\not\phi}_{\not\beta\not\alpha}\Gamma^{\not\epsilon}_{\not\phi\not\gamma}-\Gamma^{\not\phi}_{\not\beta\not\gamma}\Gamma^{\not\epsilon}_{\not\phi\not\alpha}\\
&-\Gamma^{\not\epsilon}_{\not\alpha\not\gamma\,,\not\beta}+\Gamma^{\not\epsilon}_{\not\gamma\not\beta\,,\not\alpha}+\Gamma^{\not\phi}_{\not\beta\not\gamma}\Gamma^{\not\epsilon}_{\not\phi\not\alpha}-\Gamma^{\not\phi}_{\not\gamma\not\alpha}\Gamma^{\not\epsilon}_{\not\phi\not\beta}\\
=\ &0
\end{aligned}
\tag{D3}
$$

and the same identity follows immediately for the barred form. The 'curly-hat' form is defined as;

$$
\hat{\mathcal{R}}^{\not\alpha}_{\not\alpha\not\beta\not\gamma}=\Gamma^{\not\phi}_{\not\gamma\not\alpha}\Gamma^{\not\alpha}_{\not\phi\not\beta}+\Gamma^{\not\phi}_{\not\beta\not\alpha}\Gamma^{\not\alpha}_{\not\phi\not\gamma}
\tag{D4}
$$

and the following analogue of the Bianci identity is proven for the $\hat{\mathcal{R}}_{\not\beta\not\gamma}=\hat{\mathcal{R}}_{\not\gamma\not\beta}$ tensor;

$$
\hat{\mathcal{R}}_{\not\alpha\not\beta\,;\,\not\gamma}+\hat{\mathcal{R}}_{\not\beta\not\gamma\,;\,\not\alpha}+\hat{\mathcal{R}}_{\not\gamma\not\alpha\,;\,\not\beta}=0
\tag{D5}
$$

To prove this identity we first need the identity;

$$
u_{\not\alpha\,;\,\not\beta\,;\,\not\gamma\,;\,\not\delta}+u_{\not\alpha\,;\,\not\beta\,;\,\not\delta\,;\,\not\gamma}=\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\gamma\not\delta}u_{\not\epsilon\,;\,\not\beta}+\mathcal{R}^{\not\epsilon}_{\not\beta\not\gamma\not\delta}u_{\not\alpha\,;\,\not\epsilon}
\tag{D6}
$$

where;

$$
\bar{\mathcal{R}}^{\not\phi}_{\not\alpha\not\beta\not\gamma}=-\Gamma^{\not\phi}_{\not\alpha\not\beta\,,\not\gamma}-\Gamma^{\not\phi}_{\not\alpha\not\gamma\,,\not\beta}+\Gamma^{\not\phi}_{\not\gamma\not\alpha}\Gamma^{\not\phi}_{\not\phi\not\beta}+\Gamma^{\not\phi}_{\not\beta\not\alpha}\Gamma^{\not\phi}_{\not\phi\not\gamma}
\tag{D7}
$$

the proof of which is lengthy and will not be reproduced here. It then follows that;

$$
\begin{aligned}
&(\ u_{\not\alpha\,;\,\not\beta\,;\,\not\gamma\,;\,\not\delta}+u_{\not\alpha\,;\,\not\beta\,;\,\not\delta\,;\,\not\gamma})+(u_{\not\alpha\,;\,\not\delta\,;\,\not\beta\,;\,\not\gamma}+u_{\not\alpha\,;\,\not\delta\,;\,\not\gamma\,;\,\not\beta})+(u_{\not\alpha\,;\,\not\gamma\,;\,\not\delta\,;\,\not\beta}+u_{\not\alpha\,;\,\not\gamma\,;\,\not\beta\,;\,\not\delta})\\
&=\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\gamma\not\delta}u_{\not\epsilon\,;\,\not\beta}+\mathcal{R}^{\not\epsilon}_{\not\beta\not\gamma\not\delta}u_{\not\alpha\,;\,\not\epsilon}+\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma}u_{\not\epsilon\,;\,\not\delta}+\mathcal{R}^{\not\epsilon}_{\not\delta\not\beta\not\gamma}u_{\not\alpha\,;\,\not\epsilon}+\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\delta\not\beta}u_{\not\epsilon\,;\,\not\gamma}+\mathcal{R}^{\not\epsilon}_{\not\gamma\not\delta\not\beta}u_{\not\alpha\,;\,\not\epsilon}\\
&=\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\gamma\not\delta}u_{\not\epsilon\,;\,\not\beta}+\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma}u_{\not\epsilon\,;\,\not\delta}+\bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\delta\not\beta}u_{\not\epsilon\,;\,\not\gamma}
\end{aligned}
\tag{D8}
$$

viz eq.(D5) whilst the first line of eq.(D8) can alternatively be written as;

$$
\begin{aligned}
&(\ \mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma}u_{\not\epsilon}\,)_{\,;\,\not\delta}+(\ \mathcal{R}^{\not\epsilon}_{\not\alpha\not\gamma\not\delta}u_{\not\epsilon}\,)_{\,;\,\not\beta}+(\ \mathcal{R}^{\not\epsilon}_{\not\alpha\not\delta\not\beta}u_{\not\epsilon}\,)_{\,;\,\not\gamma}\\
&=-(\mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma\,;\,\not\delta}+\mathcal{R}^{\not\epsilon}_{\not\alpha\not\gamma\not\delta\,;\,\not\beta}+\mathcal{R}^{\not\epsilon}_{\not\alpha\not\delta\not\beta\,;\,\not\gamma})\,u_{\not\epsilon}+\mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma}u_{\not\epsilon\,;\,\not\delta}+\mathcal{R}^{\not\epsilon}_{\not\alpha\not\gamma\not\delta}u_{\not\epsilon\,;\,\not\beta}+\mathcal{R}^{\not\epsilon}_{\not\alpha\not\delta\not\beta}u_{\not\epsilon\,;\,\not\gamma}
\end{aligned}
\tag{D9}
$$

so we must have;

$$\mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma\,;\,\not\delta} + \mathcal{R}^{\not\epsilon}_{\not\alpha\not\gamma\not\delta\,;\,\not\beta} + \mathcal{R}^{\not\epsilon}_{\not\alpha\not\delta\not\beta\,;\,\not\gamma} = (\mathcal{R}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma} - \bar{\mathcal{R}}^{\not\epsilon}_{\not\alpha\not\beta\not\gamma})u_{\not\epsilon\,;\,\not\delta} + \text{plus cyclic permutations.} \tag{D10}$$

Now since the connection is antisymmetric in its lower two indices when the derivative of the symmetric part of the metric vanishes (the case under consideration here);

$$\begin{aligned}\Gamma^{\not\alpha}_{\not\alpha\not\beta} &= \frac{1}{2}(g_{\not\epsilon\not\alpha\,,\,\not\beta} + g_{\not\beta\not\epsilon,\,\not\alpha} - g_{\not\alpha\not\beta\,,\,\not\epsilon})\bar{g}^{\not\epsilon\not\alpha} \\ &= \frac{1}{2}(g_{\not\epsilon}{}^{\not\alpha}{}_{,\,\not\beta} + g_{\not\beta\not\epsilon,}{}^{\not\alpha} - g^{\not\alpha}{}_{\not\beta\,,\,\not\epsilon})\bar{g}^{\not\epsilon}{}_{\not\alpha} \\ &= \frac{-1}{2}(g_{\not\epsilon}{}^{\not\alpha}{}_{,\,\not\alpha} + g_{\not\alpha\not\epsilon,}{}^{\not\alpha} - g^{\not\alpha}{}_{\not\alpha\,,\,\not\epsilon})\bar{g}^{\not\epsilon}{}_{\not\beta} \\ &= 0\end{aligned} \tag{D11}$$

so that;

$$\mathcal{R}^{\not\alpha}_{\not\alpha\not\beta\not\gamma} - \bar{\mathcal{R}}^{\not\alpha}_{\not\alpha\not\beta\not\gamma} = 2\Gamma^{\not\alpha}_{\not\alpha\not\gamma\,,\,\not\beta} + 2\Gamma^{\not\alpha}_{\not\alpha\not\beta\,,\,\not\gamma} = 0 \tag{D12}$$

and hence we finally obtain from eq.(D10);

$$\hat{\mathcal{R}}_{\not\beta\not\gamma\,;\,\not\delta} + \hat{\mathcal{R}}_{\not\gamma\not\delta\,;\,\not\beta} + \hat{\mathcal{R}}_{\not\delta\not\beta\,;\,\not\gamma} = 0 \tag{D13}$$

because the u_α is arbitrary. The minus sign in the last line of eq.(D9) arises because the vector field in question, here $u_{\not\alpha}$, must satisfy the Lörentz condition. This follows automatically since for the anti-symmetric part of the metric which contains the potential of the electro-magnetic field and has non-vanishing derivatives;

$$\Gamma^{\not\alpha}_{\not\alpha\not\beta} = 0 \;\text{ iff }\; g^{a}_{\not\alpha\not\beta}{}^{,\,\not\alpha} = 0 \Rightarrow \partial_\alpha A^\alpha = 0 \tag{D14}$$

because the arbitrary β index will be eliminated in the gauge invariant terms which appear in the stress-energy tensor, so that commuting the derivative index $\not\beta$ past the vector index $\not\alpha$ generates a minus sign plus a $2\eta_{\alpha\beta}\partial^{\not\beta}u_{\not\beta} = 0$ i.e. anti-commutes past the vector index).

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